ON SEQUENTIAL CONTINUITY OF COMPOSITION MAPPING

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Abstract. In [1] there was proved a theorem concerning the continuity of the composition mapping, and there was announced a theorem on sequential continuity of this mapping. The proof of the last theorem has not been published as yet. We prove here a more general theorem and give some corollaries. One of these corollaries is a result that was formulated by Lang in [2] as a conjecture.

0. Introduction

In [1] there was proved a theorem concerning the continuity of the composition mapping which sends a pair of functions between (pseudo)topological vector spaces to their composition, and there was announced a theorem on sequential continuity of this mapping for functions between topological vector spaces. The proof of the last theorem has not been published as yet. We prove here a more general theorem and give some corollaries. One of these corollaries is a result that was formulated by Lang in [2] as a conjecture.

1. Definitions and notations

Let X be a topological vector space. By $\mathrm{Nb}_0(X)$ we denote the set of all neighbourhoods of zero in X.

Definition 1. A set $A \subset X$ is *bounded* if for every $U \in \mathrm{Nb}_0(X)$ there exists $\delta > 0$ such that $\delta A \subset U$.

Definition 2. A set $A \subset X$ is *sequentially compact* if we can choose from every sequence of its elements a converging subsequence.

Definition 3. Let X, Y be topological spaces, and let f be a mapping from X into Y. We say that f is *sequentially continuous at a point* x if for any sequence $\{x_n\}$ in X

$$x_n \to x \text{ in } X \text{ implies } f(x_n) \to f(x) \text{ in } Y.$$

Definition 4. Let X, Y be topological vector spaces, and let f be a mapping from X into Y. We say that f is *uniformly sequentially continuous on a set* $A \subset X$ if

$$\forall \left(\{h_n\} \subset X \ , h_n \xrightarrow[n\to\infty]{} 0\right) \ \forall V \in \mathrm{Nb}_0(Y) \ \exists n_0 \in \mathbb{N} \ \forall n \geq n_0 \forall x \in A :$$

$$f(x+h_n) - f(x) \in V.$$

Definition 5. Let $\mathcal{F}(X,Y)$ be the vector space of all mappings from a topological vector space X into a topological vector space Y. Let $\mathcal{S}$ be a system of subsets in X defined in terms of the topology of X (e.g. the system of all bounded sets or the system of all sequentially compact sets in X). *The topology of uniform convergence on the system* $\mathcal{S}$ is the topology with the following base $\mathcal{B}$ of neighbourhoods of zero:

$$\mathcal{B} = \{U_{A,V} \mid A \in \mathcal{S}, V \in \mathrm{Nb}_0(Y)\},$$

where $U_{A,V} := \{f \in \mathcal{F}(X,Y) \mid f(A) \subset V\}$.
Here $f(A) = \{f(x)|x \in A\}$.

This topology is denoted by $\mathcal{F}_{\mathcal{S}}$, the topological vector space with this topology being denoted by $\mathcal{F}_{\mathcal{S}}(X,Y)$.

Definition 6. Let X, Y, Z be topological vector spaces. *The composition mapping* is the mapping

$$\mathrm{comp}: \quad \mathcal{F}(X,Y) \times \mathcal{F}(Y,Z) \to \mathcal{F}(X,Z), \quad (f,g) \mapsto g \circ f.$$

Definition 7([3])**.** Let X be a topological space.

A set $U \subset X$ is *sequentially open* if for every sequence $\{x_n\} \subset X$ converging to a point $x \in U$ there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0 :$ $x_n \in U$.

A set $V \subset X$ is *sequentially closed* if, whenever $\{x_n\}$ is a sequence in V converging to x, then x must also lie in V.

Remark. It is obvious that every open subset of X is sequentially open and every closed subset of X is sequentially closed.

Definition 8([3])**.** A topological space X is a *sequential space* if one of the following equivalent conditions is satisfied:

(1) Every sequentially open subset of X is open.
(2) Every sequentially closed subset of X is closed.

Note that every metrizable space is a sequential space.

2. Sequential continuity of composition mapping

Lemma 1. *Let X be a set, Y a topological vector space, $\mathcal{S}$ a set of sequences in X, and $\{f_n\}$ a sequence of mappings from X into Y. If $f_n(x_n) \to 0$ for all sequences $\{x_n\}$ from $\mathcal{S}$, then the sequence $\{f_n\}$ converges to zero in Y uniformly on every set in X such that from every sequence of its elements we can choose a subsequence which lies in $\mathcal{S}$.*

Proof. Let A be a subset in X such that if $\{x_n\} \subset A$; then there exists a subseqeunce $\{x_{n_i}\}$ which lies in $\mathcal{S}$. Let us suppose that there exists a neighbourhood of zero V in Y such that $\forall i \in \mathbb{N}\ \exists\, (n_i \in \mathbb{N}, n_i \geq i) :$ $f_{n_i}(A) \not\subset V$. Then there exists $x_{n_i} \in A$ such that $f_{n_i}(x_{n_i}) \notin V$. Choose a subsequence $\{x_{n_{i_k}}\} \in \mathcal{S}$. For any k it holds $f_{n_{i_k}}(x_{n_{i_k}}) \notin V$. Hence

$f_{n_{i_k}}(x_{n_{i_k}})$ does not converge to zero in Y, and neither does the sequence $\{f_n(x_n)\}$. We come to a contradiction to our assumption, and the theorem is proved.

Theorem 1. *Let for each topological vector space E a system $\mathcal{S}(E)$ of subsets of E be given that satisfies the conditions:*

(i) $\mathcal{S}(E)$ *contains all converging sequences in* E,
(ii) *if* $A, B \in \mathcal{S}(E)$ *then* $A + B \in \mathcal{S}(E)$, (where $A + B = \{a + b \mid a \in A, b \in B\}$),
(iii) *if* $A \in \mathcal{S}(E)$ *and* $B \subset A$ *then* $B \in \mathcal{S}(E)$.

Let X, Y, Z be topological vector spaces. If a mapping $f : X \to Y$ sends the sets from $\mathcal{S}(X)$ into sets from $\mathcal{S}(Y)$ and a mapping $g : Y \to Z$ is uniformly sequentially continuous on the sets from $\mathcal{S}(Y)$, then the composition mapping (see Def. 6) is sequentially continuous at the point (f, g).

Proof. For brevity we write in the text below simply $\mathcal{F}_{\mathcal{S}}$ instead of $\mathcal{F}_{\mathcal{S}(X)}$.

Let $\widetilde{\mathcal{S}}(X)$ denote the system of all sequences that lie in $\mathcal{S}(X)$. Let $\{f_n\}$ be a sequence of mappings from X into Y that converges to zero in $\mathcal{F}_{\mathcal{S}}$ and let $\{g_n\}$ be a sequence of mappings from Y into Z that converges to zero in $\mathcal{F}_{\mathcal{S}}$. It is clear that the sequences $\{f_n\}$ and $\{g_n\}$ converge to zero in $\mathcal{F}_{\widetilde{\mathcal{S}}}$ too.

We have to show that the sequence $\{\mathrm{comp}(f + f_n, g + g_n)\}$ converges to $\mathrm{comp}(f, g)$ in $\mathcal{F}_{\mathcal{S}}$. By Lemma 1 it is sufficient to show that for all sequences $\{x_n\}$ from $\widetilde{\mathcal{S}}(X)$ it holds

$$((g + g_n)((f + f_n)(x_n)) - (g(f(x_n))) \xrightarrow[n\to\infty]{} 0 \text{ in } Z,$$

that is, that

$$(g(f(x_n) + f_n(x_n)) + g_n(f(x_n) + f_n(x_n)) - g(f(x_n))) \xrightarrow[n\to\infty]{} 0 \text{ in } Z,$$

or, in terms of neighbourhoods, that

$$\forall W \in \mathrm{Nb}_0(Z)\ \exists n_0 \in \mathbb{N} \quad \forall n \geq n_0 :$$

$$(g(f(x_n) + f_n(x_n)) + g_n(f(x_n) + f_n(x_n)) - g(f(x_n))) \in W.$$

Without loss of generality we can assume that W is balanced. There exists a balanced neighbourhood of zero V in Z such that $V + V \subset W$.

Since $\{f(x_n)\} \in \widetilde{\mathcal{S}}(Y)$, $f_n(x_n) \xrightarrow[n\to\infty]{} 0$ and g is uniformly sequentially continuous on the sets from $\widetilde{\mathcal{S}}(Y)$ we have

$$\exists n_1 \in \mathbb{N}\ \forall n \geq n_1 \forall k \in \mathbb{N} :\ (g(f(x_k) + f_n(x_n)) - g(f(x_k))) \in V.$$

Since the sequence $\{f_n(x_n)\}$ converges to zero and therefore belongs (as a set) to $\widetilde{\mathcal{S}}(Y)$, by (i), the sequence $\{f(x_n) + f_n(x_n)\}$ belongs as a set to $\mathcal{S}(Y)$, by (ii) and (iii). Since $g_n \xrightarrow[n\to\infty]{} 0$ in $\mathcal{F}_{\mathcal{S}}$, we conclude that

$$\exists n_2 \in \mathbb{N}\ \forall n \geq n_2\ : g_n(f(x_n) + f_n(x_n)) \in V.$$

Put $n_0 = \max\{n_1, n_2\}$. Then for any $n \geq n_0$ it holds

$$(g(f(x_n) + f_n(x_n)) + g_n(f(x_n) + f_n(x_n)) - g(f(x_n))) \in V + V \subset W.$$

The theorem is proved.

For any topological vector space E, let $\mathrm{B}(E)$ denote the system of all bounded sets and $\mathrm{K}(E)$ the system of all sequentially compact sets. Since the systems B and K evidently satisfy the hypothesis of Theorem 1, we obtain the following corollary, which is the result announced in [1].

Corollary 1. *Let $\mathcal{S}$ be* B *or* K, *and let X, Y, Z be topological vector spaces. If the mapping $f : X \to Y$ sends sets from $\mathcal{S}(X)$ into sets from $\mathcal{S}(Y)$ and the mapping $g : Y \to Z$ is uniformly sequentially continuous on the sets from $\mathcal{S}(Y)$, then the composition mapping is sequentially continuous at the point (f, g).*

3. The proof of a conjecture of Lang

Let X and Y be topological vector spaces. We denote the space of all continuous linear mappings from X into Y equipped with the topology of uniform convergence on all bounded subsets of X by $\mathcal{L}(X, Y)_{\mathrm{b}}$.

The following result was formulated in [2] (pgs 5-6) as a conjecture.

Theorem 2. *Let U be an open subset of a Fréchet locally convex space F, let X, Y, Z be topological vector spaces, and let the mappings $f : U \to \mathcal{L}(X, Y)_{\mathrm{b}}$ and $g : U \to \mathcal{L}(Y, Z)_{\mathrm{b}}$ be continuous. Then the mapping*

$$\varphi : U \to \mathcal{L}(X, Z)_{\mathrm{b}}, \quad x \mapsto g(x) \circ f(x)$$

is continuous.

First of all we give three lemmas and a corollary of Theorem 1.

Lemma 2. (see e.g. [4])**.** *Let (X, τ) be a sequential space, and U an open subset of X. Then the topological space (U, σ), where σ is the induced topology on U, is a sequential space.*

Lemma 3. *Let X be a sequential space, and let Y be a topological space. Let $f : X \to Y$ be a mapping. Then f is continuous if and only if f is sequentially continuous.*
Proof. See e.g. [3].

Lemma 4. *Let X, Y, Z be topological spaces. Let $f : X \to Y$ be sequentially continuous at a point $x \in X$ and let $g : Y \to Z$ be sequentially continuous at the point $f(x) \in Y$. Then $g \circ f$ is sequentially continuous at the point $x \in X$.*
Proof. Obvious.

Corollary 2. *Let U be an open subset of a sequencial topological space F, let X, Y, Z be topological vector spaces and let the mappings $f : U \to \mathcal{L}(X,Y)_{\mathrm{b}}$ and $g : U \to \mathcal{L}(Y,Z)_{\mathrm{b}}$ be continuous. Then the mapping*

$$\varphi : U \to \mathcal{L}(X,Z)_{\mathrm{b}}, \quad x \mapsto g(x) \circ f(x)$$

is continuous.

Proof. 1° First of all we claim that

$$\mathrm{comp}: \quad \mathcal{L}(X,Y)_{\mathrm{b}} \times \mathcal{L}(Y,Z)_{\mathrm{b}} \to \mathcal{L}(X,Z)_{\mathrm{b}}, \quad (l,h) \mapsto h \circ l$$

is sequentially continuous at each point (l,h).

It is sufficient to verify that continuous linear mappings satisfy the hypothesis of Theorem 1. But indeed, any continuous linear mapping sends bounded sets into bounded sets, and any continuous linear mapping h is uniformly continuous on bounded subsets, so it is evidently uniformly sequentially continuous on bounded sets.

2° Since f and g are continuous, their product

$$(f,g) : U \to \mathcal{L}(X,Y)_{\mathrm{b}} \times \mathcal{L}(Y,Z)_{\mathrm{b}}, \quad x \mapsto (f(x), g(x))$$

is continuous too. Using Lemma 2 and Lemma 3 we conclude that the mapping (f,g) is sequentially continuous.

3° Our mapping φ can be written as $\varphi = \mathrm{comp} \circ (f,g)$, therefore, by Lemma 4 and by steps 1° and 2°, the mapping φ is sequentially continuous. Using once again Lemma 3, we get that φ is continuous. The corollary is proved.

Proof of Lang's conjecture. Since every Fréchet space is first countable, and every first countable space is sequential (see e.g [5]), the assertion follows from Corollary 2.

References

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